Math 260: Linear Algebra
Proofs: Mathematical Induction

Use mathematical induction to prove …

1) $\forall$ integers $n \geq 0$, $n^3 + n$ is even.

S_n: n^3+n is even S_0: 0^3+0 is even
S_k: k^3+k is even S_{k+1}: $(k+1)^3+(k+1)$ is even.

proof:

① Base Step Since $0^3+0=0$ is even, S_0 is true ✓

② Induction step (Need $S_k \Rightarrow S_{k+1}$).

Suppose S_k is true. Then k^3+k is even, so $k^3+k=2m$ for some integer m.

$$
\begin{aligned}
\text{Then } (k+1)^3+(k+1) &= k^3+3k^2+3k+1+k+1 \\
&= \underbrace{k^3+k}_{\downarrow} + 3k^2+3k+2 \\
&= 2m + 3k(k+1) + 2
\end{aligned}
$$

case 1
If k is even, $k=2r$ for some integer r, so $3k(k+1) = 3(2r)(2r+1)$
$= 2(6r^2+3r)$
which is even

case 2
If k is odd, $k=2r+1$ for some integer r, so $3k(k+1) = 3(2r+1)(2r+2)$
$= 2(3)(2r+1)(r+1)$
which is even.

So in either case, $3k(k+1)$ is even.

So $(k+1)^3+(k+1) = 2m+3k(k+1)+2$ is the sum of 3 even numbers, so it's even.

So $S_k \Rightarrow S_{k+1}$ ✓

So by induction, $\forall$ integers $n \geq 0$, n^3+n is even.

Use mathematical induction to prove …

2) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

$S_n: \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\qquad$ $S_1: \sum_{i=1}^{1} i^2 = \frac{1(1+1)(2 \cdot 1+1)}{6}$

$S_k: \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ $\qquad$ $S_{k+1}: \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

proof:

① Base step: $\sum_{i=1}^{1} i^2 = 1^2 = 1$ and $\frac{1(1+1)(2 \cdot 1+1)}{6} = \frac{6}{6} = 1$

So $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2 \cdot 1+1)}{6} \Rightarrow S_1$ is true. ✓

② Induction step: (Need $S_k \Rightarrow S_{k+1}$)

Suppose S_k is true. That is, suppose $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$;

Then, $\sum_{i=1}^{k+1} i^2 = \left[\sum_{i=1}^{k} i^2\right] + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$

$= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

So $S_k \Rightarrow S_{k+1}$.

So by induction, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ integers $n \geq 1$.

Use mathematical induction to prove …

3) $1+3+3^2+3^3+3^4+\cdots+3^n=\frac{3^{n+1}-1}{2}$ for all integers $n \geq 1$.

S_n: $1+3+3^2+\cdots+3^n = \dfrac{3^{n+1}-1}{2}$ $\qquad$ S_1: $1+3=\dfrac{3^{1+1}-1}{2}$

S_k: $1+3+3^2+\cdots+3^k = \dfrac{3^{k+1}-1}{2}$ $\qquad$ S_{k+1}: $1+3+3^2+\cdots+3^{k+1} = \dfrac{3^{(k+1)+1}-1}{2}$

Proof:

1) Base step: Since $1+3=4$ and $\dfrac{3^{1+1}-1}{2} = \dfrac{9-1}{2} = 4$,

$1+3 = \dfrac{3^{1+1}-1}{2} \Rightarrow S_1$ is true ✓

2) Induction step: (Need $S_k \Rightarrow S_{k+1}$).

Suppose S_k is true. That is, suppose $1+3+3^2+\cdots+3^k = \dfrac{3^{k+1}-1}{2}$,

Then $1+3+3^2+\cdots+3^k+3^{k+1} = \dfrac{3^{k+1}-1}{2} + 3^{k+1}$

$= \dfrac{3^{k+1}-1}{2} + \dfrac{2\cdot 3^{k+1}}{2}$

$= \dfrac{3\cdot 3^{k+1}-1}{2} = \dfrac{3^{k+2}-1}{2}$

$= \dfrac{3^{(k+1)+1}-1}{2} \Rightarrow S_{k+1}$ is true ✓

So $S_k \Rightarrow S_{k+1}$.

So by induction, $1+3+3^2+\cdots+3^n = \dfrac{3^{n+1}-1}{2}$ $\forall$ integers $n \geq 1$.

Use mathematical induction to prove …

4) $\forall\ n \in \mathbb{N}$, $7^n - 3^n$ is a multiple of 4.

S_n: $7^n - 3^n$ is a multiple of 4

S_1: $7^1 - 3^1$ is a multiple of 4

S_K: $7^K - 3^K$ is a multiple of 4

S_{K+1}: $7^{K+1} - 3^{K+1}$ is a multiple of 4.

proof:

① Base step — Since $7^1 - 3^1 = 4$ and 4 is a multiple of 4,
S_1 is true ✓

② Induction step (Need $S_K \Rightarrow S_{K+1}$).

Suppose S_K is true. That is, suppose $7^K - 3^K$ is a multiple of 4.
Then $\exists$ an integer m such that $7^K - 3^K = 4m$. $(7^K = 4m + 3^K)$

Then, $7^{K+1} - 3^{K+1} = 7 \cdot 7^K - 3^{K+1} = 7 \cdot (4m + 3^K) - 3^{K+1}$

$= 28m + 7 \cdot 3^K - 3 \cdot 3^K = 28m + 4 \cdot 3^K$

$= 4(7m + 3^K) = 4(\text{integer})$

So $7^{K+1} - 3^{K+1}$ is a multiple of 4. So S_{K+1} is true.

So $S_K \Rightarrow S_{K+1}$ ✓

So by induction, $7^n - 3^n$ is a multiple of 4 for all integers $n \geq 1$.